

Note on the $N = 2$ super Yang-Mills gauge theory in a noncommutative differential geometry

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Abstract. The $N = 2$ super-Yang-Mills gauge theory is reconstructed in a non-commutative differential geometry (NCG). Our NCG with one-form bases dx^μ on the Minkowski space M_4 and χ on the discrete space Z_2 is a generalization of the ordinary differential geometry on the continuous manifold. Thus, the generalized gauge field is written as $\mathcal{A}(x, y) = A_\mu(x, y)dx^\mu + \Phi(x, y)\chi$ where y is the argument in Z_2 . $\Phi(x, y)$ corresponds to the scalar and pseudo-scalar bosons in the $N = 2$ super Yang-Mills gauge theory whereas it corresponds to the Higgs field in the ordinary spontaneously broken gauge theory. Using the generalized field strength constructed from $\mathcal{A}(x, y)$ we can obtain the bosonic Lagrangian of the $N = 2$ super Yang-Mills gauge theory in the same way as Chamseddine first introduced the supersymmetric Lagrangian of the $N = 2$ and $N = 4$ super Yang-Mills gauge theories within the framework of Connes's NCG. The fermionic sector is introduced so as to satisfy the invariance of the total Lagrangian with respect to supersymmetry.

1 Introduction

Since Connes [1] proposed the original idea concerning the reconstruction of spontaneously broken gauge theories by use of noncommutative differential geometry (NCG) on the discrete space, many works [2–11] have appeared in order to realize the unified picture of gauge and Higgs fields as the generalized gauge field on the discrete space $M_4 \times Z_2$. Especially, the Standard Model has successfully reconstructed in various versions of NCG.

We have also proposed a characteristic formulation [8]–[11] which is the generalization of the usual differential geometry on an ordinary manifold into the discrete space $M_4 \times Z_N$. In a noncommutative geometry on $M_4 \times Z_2$, the extra differential one-form χ is introduced in addition to the usual one-form dx^μ and therefore, our formulation is very similar to the ordinary differential geometry. Our formulation includes the symmetry breaking matrix, so that it is flexible enough to enable us to reconstruct not only the Standard Model but the gauge theories with complex symmetry breaking structures such as the SU(5) GUT [9] and the SO(10) GUT [10].

On the other hand, the supersymmetric gauge theories including the minimal supersymmetric Standard Model (MSSM), and the supersymmetric SU(5) and SO(10) GUTs have the many attractive features as already known. Then, it is tempting to reconstruct supersymmetric Yang-Mills gauge theory in NCG on the super-manifold with the arguments x_μ , θ_A and $\bar{\theta}^{\dot{A}}$ multiplying the discrete space Z_2 . However, this approach is somewhat difficult to be realized as pointed by Chamseddine [4]. He rather took an approach to reconstruct the supersymmetric Lagrangian within the ordinary framework of NCG. He succeeded in

obtaining the supersymmetric Lagrangian of the $N = 2$ and $N = 4$ super Yang-Mills gauge theories [4]. In this paper, following to Chamseddine's idea, we try to reconstruct the $N = 2$ super Yang-Mills gauge theory based on our formulation of NCG. In second section, our NCG scheme on the discrete space $M_4 \times Z_2$ is reviewed because it has not been well-known among particle physicists. In third section, we aim to construct the following Lagrangian of the $N = 2$ super Yang-Mills gauge theory.

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_D, \quad (1)$$

where

$$\mathcal{L}_B = \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu}^\dagger(x) F^{\mu\nu}(x) + \frac{1}{2} [D_\mu S(x)]^2 + [D_\mu P(x)]^2 + \frac{g^2}{2} [S(x), P(x)]^2 \right\}, \quad (2)$$

$$\mathcal{L}_D = \text{Tr} \left\{ i\bar{\psi}(x)\gamma^\mu D_\mu \psi(x) + \bar{\psi}(x) \times [S(x) + i\gamma^5 P(x), \psi(x)] \right\}. \quad (3)$$

Here, $S(x)$ and $P(x)$ are scalar and pseudo-scalar fields, and $\psi(x)$ is a Dirac spinor, all in the adjoint representation of the internal gauge group, and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\mu(x), A_\nu(x)], \quad (4)$$

$$D_\mu S(x) = \partial_\mu S(x) - ig[A_\mu(x), S(x)], \quad (5)$$

$$D_\mu P(x) = \partial_\mu P(x) - ig[A_\mu(x), P(x)], \quad (6)$$

$$D_\mu \psi(x) = \partial_\mu \psi(x) - ig[A_\mu(x), \psi(x)] \quad (7)$$

with the super Yang-Mills gauge field $A_\mu(x)$. Fourth section is devoted to concluding remarks.

2 Our NCG formulation

First, let us briefly present our formulation of NCG by modifying it so as to be applicable to the super Yang-Mills gauge theory. The generalized gauge field $\mathcal{A}(x, y)$ on the discrete space $M_4 \times Z_2$ is introduced as

$$\mathcal{A}(x, y) = \sum_i a_i^\dagger(x, y) \mathbf{d}a_i(x, y), \quad (8)$$

where $a_i(x, y)$ is the square-matrix-valued functions. Here, the subscript i is a variable corresponding to the extra internal space which we can not yet identify. At this time, we simply regard $a_i(x, y)$ as the more fundamental field to construct gauge and Higgs fields. y is an argument in Z_2 and takes the value $y = +$ or $y = -$. The operator \mathbf{d} in (8) is the generalized exterior derivative defined as follows.

$$\mathbf{d} = d + d_\chi, \quad (9)$$

$$da_i(x, y) = \partial_\mu a_i(x, y) dx^\mu, \quad (10)$$

$$d_\chi a_i(x, y) = \partial_y a_i(x, y) \chi \\ = [-a_i(x, y)M(y) + M(y)a_i(x, -y)]\chi, \quad (11)$$

where dx^μ is ordinary one-form basis, taken to be dimensionless, in Minkowski space M_4 , and χ is the one-form basis, assumed to be also dimensionless, in the discrete space Z_2 . The operator ∂_y defined in (11) is a difference operator accompanied by the x -independent matrix $M(y)$ whose hermitian conjugation is given by $M(y)^\dagger = M(-y)$. The introduction of $M(y)$ is needed to assure the consistent sum of matrices on the right-hand side of (11). In order to find the explicit forms of gauge and Higgs fields according to (8)–(11), we need the following important algebraic rule of a noncommutative geometry:

$$f(x, y)\chi = \chi f(x, -y), \quad (12)$$

where $f(x, y)$ is a field $a_i(x, y)$, gauge field, Higgs field or fermion field defined on the discrete space. It should be noted that (12) does not express the relation between the matrix elements of $f(x, +)$ and $f(x, -)$ but insures the consistent product between the fields expressed in differential form on the discrete space. Inserting (9)–(11) into (8) and using (12), $\mathcal{A}(x, y)$ is rewritten as

$$\mathcal{A}(x, y) = A_\mu(x, y)dx^\mu + \Phi(x, y)\chi, \quad (13)$$

where

$$A_\mu(x, y) = \sum_i a_i^\dagger(x, y) \partial_\mu a_i(x, y), \quad (14)$$

$$\Phi(x, y) = \sum_i a_i^\dagger(x, y) (-a_i(x, y)M(y) \\ + M(y)a_i(x, -y)). \quad (15)$$

Here, $A_\mu(x, y)$ and $\Phi(x, y)$ are identified with the gauge field in the flavor symmetry and the Higgs field, respectively. In order to identify $A_\mu(x, y)$ with a true gauge field, the following conditions must be imposed.

$$\sum_i a_i^\dagger(x, y) a_i(x, y) = 1. \quad (16)$$

Before constructing the gauge covariant field strength, we address the gauge transformation of $a_i(x, y)$ which is defined as

$$a_i^g(x, y) = a_i(x, y)g(x, y), \quad (17)$$

where $g(x, y)$ is the gauge function with respect to the corresponding flavor unitary group. We will take $g(x, +) = g(x, -) = g(x)$ in this article because all fields under considerations in (2) and (3) belong to the adjoint representation. However, the argument y in $g(x, y)$ is kept in equations for a while. Then, we can find from (8) and (17) the gauge transformation of $\mathcal{A}(x, y)$ to be

$$\mathcal{A}^g(x, y) = g^{-1}(x, y)\mathcal{A}(x, y)g(x, y) + g^{-1}(x, y)\mathbf{d}g(x, y), \quad (18)$$

where as in (9)–(11),

$$\mathbf{d}g(x, y) = (d + d_\chi)g(x, y) = \partial_\mu g(x, y)dx^\mu + \partial_y g(x, y)\chi \\ = \partial_\mu g(x, y)dx^\mu \\ + [-g(x, y)M(y) + M(y)g(x, -y)]\chi. \quad (19)$$

Using (17) and (18), we can find the gauge transformations of gauge and Higgs fields as

$$A_\mu^g(x, y) = g^{-1}(x, y)A_\mu(x, y)g(x, y) \\ + g^{-1}(x, y)\partial_\mu g(x, y), \quad (20)$$

$$\Phi^g(x, y) = g^{-1}(x, y)\Phi(x, y)g(x, -y) \\ + g^{-1}(x, y)\partial_y g(x, y), \quad (21)$$

Equation (21) is very similar to (20) that is the gauge transformation of the genuine gauge field $A_\mu(x, y)$ and therefore it strongly indicates that the Higgs field is a kind of gauge field on the discrete space $M_4 \times Z_2$. From (19), (21) is rewritten as

$$\Phi^g(x, y) + M(y) = g^{-1}(x, y)(\Phi(x, y) + M(y))g(x, -y). \quad (22)$$

Here, we define the field $H(x, y)$ as

$$H(x, y) = \Phi(x, y) + M(y). \quad (23)$$

From (22), it seems that $H(x, y)$ is the un-shifted Higgs field because we assume that $M(y)$ is invariant against the gauge transformation. Therefore, if $\Phi(x, y)$ has a vanishing vacuum expectation value, $M(y)$ is identified with the vacuum expectation value of $H(x, y)$. However, in this article, we do not determine what value of the vacuum expectation $\Phi(x, y)$ takes. Therefore, there is a possibility that $\Phi(x, y)$ has the non-vanishing vacuum expectation value. We must adopt this case to reconstruct the super Yang-Mills gauge theory, as shown later.

In addition to the algebraic rules in (9)–(11) we add one more important rule that

$$d_\chi M(y) = M(y)M(-y)\chi \quad (24)$$

which together with (11) yields the nilpotency of d_χ and then the nilpotency of the generalized exterior derivative \mathbf{d} . For the proof of nilpotency of d_χ , see [8]. With these

considerations we can construct the gauge covariant field strength as follows:

$$\mathcal{F}(x, y) = d\mathcal{A}(x, y) + \mathcal{A}(x, y) \wedge \mathcal{A}(x, y). \quad (25)$$

From (18) and (25) we can easily find the gauge transformation of $\mathcal{F}(x, y)$ as

$$\mathcal{F}^g(x, y) = g^{-1}(x, y)\mathcal{F}(x, y)g(x, y). \quad (26)$$

The bosonic Lagrangian is obtained by

$$\mathcal{L}_B = -\frac{1}{4g^2} \sum_{y=\pm} \text{Tr} \langle \mathcal{F}(x, y), \mathcal{F}(x, y) \rangle \quad (27)$$

where g is a constant relating to the coupling constant of the flavor gauge field and Tr denotes the trace over internal symmetry matrices. In order to express the bosonic Lagrangian let us denote the explicit expressions of the field strength $\mathcal{F}(x, y)$. The algebraic rules defined in (9)–(11), (12) and (16) yield

$$\begin{aligned} \mathcal{F}(x, y) &= \frac{1}{2} F_{\mu\nu}(x, y) dx^\mu \wedge dx^\nu \\ &+ D_\mu H(x, y) dx^\mu \wedge \chi + V(x, y) \chi \wedge \chi, \end{aligned} \quad (28)$$

where

$$\begin{aligned} F_{\mu\nu}(x, y) &= \partial_\mu A_\nu(x, y) - \partial_\nu A_\mu(x, y) \\ &+ [A_\mu(x, y), A_\nu(x, y)], \end{aligned} \quad (29)$$

$$\begin{aligned} D_\mu H(x, y) &= \partial_\mu H(x, y) + A_\mu(x, y)H(x, y) \\ &- H(x, y)A_\mu(x, -y), \end{aligned} \quad (30)$$

$$V(x, y) = H(x, y)H(x, -y) - Y(x, y). \quad (31)$$

The quantity $Y(x, y)$ in (31) is an auxiliary field and expressed as

$$Y(x, y) = \sum_i a_i^\dagger(x, y)M(y)M(-y)a_i(x, y), \quad (32)$$

which in general may or may not depend on $\Phi(x, y)$ and/or may be a constant field. However, in the case of the super Yang-Mills gauge theory, $Y(x, y)$ must be an independent auxiliary field.

In order to obtain the explicit expression of L_B in (27) we must determine the metric structure of one-forms.

$$\begin{aligned} \langle dx^\mu, dx^\nu \rangle &= g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(1, -1, -1, -1), \\ \langle \chi, \chi \rangle &= -1, \quad \langle dx^\mu, \chi \rangle = 0. \end{aligned} \quad (33)$$

From (28)–(31), \mathcal{L}_B is written as

$$\begin{aligned} \mathcal{L}_B &= -\frac{1}{4} \text{Tr} \sum_{y=\pm} \frac{1}{2g^2} F_{\mu\nu}^\dagger(x, y) F^{\mu\nu}(x, y) \\ &+ \frac{1}{4} \text{Tr} \sum_{y=\pm} \frac{1}{g^2} D_\mu H(x, y)^\dagger D^\mu H(x, y) \\ &- \frac{1}{4} \text{Tr} \sum_{y=\pm} \frac{1}{g^2} V^\dagger(x, y) V(x, y), \end{aligned} \quad (34)$$

where the third term on the right hand side of (34) is the potential term of Higgs particle.

3 Super Yang-Mills gauge theory

In the case of the $N = 2$ super Yang-Mills theory we must assume that $A_\mu(x, +) = A_\mu(x, -) = -igA_\mu(x)$ and $H(x, +) = H^\dagger(x, -) = gH(x)$. In addition, $Y(x, \pm)$ are independent auxiliary fields and satisfy the condition $Y(x, +) = Y(x, -) = Y(x)$ which implies $[M(+), M(-)] = 0$ as pointed out by Chamseddine [4]. In the reconstruction of the $N = 2$ super Yang-Mills gauge theory, we must consider that $M(y)$ is not necessarily vacuum expectation value of the Higgs field $H(x, y)$ in (23). This case is allowable in our formulation. After the elimination of the auxiliary field $Y(x)$ owing to the equation of motion, these assumptions yield that

$$\begin{aligned} \mathcal{L}_B &= \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \right. \\ &\left. + \frac{1}{2} D_\mu H(x)^\dagger D^\mu H(x) - \frac{g^2}{8} V(x)^2 \right\}, \end{aligned} \quad (35)$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \quad (36)$$

$$D^\mu H(x) = \partial_\mu H(x) - ig[A_\mu, H(x)], \quad (37)$$

$$V(x) = [H(x), H^\dagger(x)]. \quad (38)$$

If $H(x) = S(x) - iP(x)$, (35) is readily found to be the Lagrangian of the $N = 2$ super Yang-Mills gauge theory given in (2). The more explicit specifications of the corresponding fields are given as follows:

$$A_\mu(x) = \sum_a T^a A_\mu^a(x), \quad (39)$$

$$H(x) = \sum_a T^a (S^a(x) - iP^a(x)), \quad (40)$$

where T^a are matrices of internal symmetry with the orthogonal condition $\text{Tr}(T^a T^b) = \delta^{ab}$ and the commutation relation $[T^a, T^b] = if^{abc}T^c$. Inserting (39) and (40) into (35), we find

$$\begin{aligned} \mathcal{L}_B &= \sum_a \left\{ -\frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} (D_\mu S^a)^2 \right. \\ &\left. + \frac{1}{2} (D_\mu P^a)^2 - \frac{g^2}{2} (f^{abc} S^b P^c)^2 \right\}, \end{aligned} \quad (41)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \quad (42)$$

$$D_\mu S^a = \partial_\mu S^a + gf^{abc} A_\mu^b S^c. \quad (43)$$

$$D_\mu P^a = \partial_\mu P^a + gf^{abc} A_\mu^b P^c. \quad (44)$$

Let us turn to the fermion sector and briefly review the way to construct the Dirac Lagrangian in (3). The covariant derivative acting on the spinor field $\psi(x, y)$ which

is the adjoint representation of the flavor gauge group is defined as

$$\mathcal{D}\psi(x, y) = [\mathbf{d} + \mathcal{A}(x, y), \psi(x, y)], \quad (45)$$

which we call the covariant spinor one-form. This expression is due to the fact that $\psi(x, y)$ belongs to the adjoint representation of the flavour symmetry. Since the role of d_χ makes the shift $\Phi(x, y) \rightarrow \Phi(x, y) + M(y) = H(x, y)$ as shown previously, we define also for fermion field

$$[d_\chi, \psi(x, y)] = [M(y), \psi(x, -y)]\chi \quad (46)$$

which leads (45) to

$$\begin{aligned} \mathcal{D}\psi(x, y) = & (\partial_\mu \psi(x, y) + [A_\mu(x, y), \psi(x, y)])dx^\mu \\ & + [H(x, y), \psi(x, -y)]\chi. \end{aligned} \quad (47)$$

In deriving (47), use has been made of (12). As $\psi(x, y)$ is subjected to the gauge transformation

$$\psi^g(x, y) = g^{-1}(x)\psi(x, y)g(x), \quad (48)$$

$\mathcal{D}\psi(x, y)$ becomes gauge covariant owing to (20), (22) and (47):

$$\mathcal{D}\psi^g(x, y) = g^{-1}(x)\mathcal{D}\psi(x, y)g(x). \quad (49)$$

In addition, $\mathbf{d} + \mathcal{A}(x, y)$ is Lorentz invariant, and so $\mathcal{D}\psi(x, y)$ is transformed as a spinor just like $\psi(x, y)$ against Lorentz transformation.

In order to obtain the Dirac Lagrangian for fermion sector, the associated spinor one-form is introduced as the counter-part of (45) by

$$\tilde{\mathcal{D}}\psi(x, y) = \gamma_\mu \psi(x, y)dx^\mu - i\psi(x, y)\chi. \quad (50)$$

With the same inner products for spinor one-forms as in [8] that

$$\begin{aligned} \langle A(x, y)dx^\mu, B(x, y)dx^\nu \rangle &= \bar{A}(x, y)B(x, y)g^{\mu\nu}, \\ \langle A(x, y)\chi, B(x, y)\chi \rangle &= -\bar{A}(x, y)B(x, y), \end{aligned} \quad (51)$$

and vanishing other such combinations, we can obtain the Dirac Lagrangian

$$\begin{aligned} \mathcal{L}_D(x, y) &= i\text{Tr} \langle \tilde{\mathcal{D}}\psi(x, y), \mathcal{D}\psi(x, y) \rangle \\ &= \text{Tr} \{ i [\bar{\psi}(x, y)\gamma^\mu (\partial_\mu \psi(x, y) + [A_\mu(x, y), \psi(x, y)]) \\ &\quad + \bar{\psi}(x, y)[H(x, y), \psi(x, -y)]] \}, \end{aligned} \quad (52)$$

where Tr is also the trace over internal symmetry matrices. In this article, we specify $\psi(x, +) = \psi(x)$ and $\psi(x, -) = \psi(x)$. Then, the total Dirac Lagrangian is obtained by summing (52) over y :

$$\begin{aligned} \mathcal{L}_D &= \sum_{y=\pm} \mathcal{L}_D(x, y), \\ &= \text{Tr} \{ i\bar{\psi}(x)\gamma^\mu (\partial_\mu \psi(x) - ig[A_\mu(x), \psi(x)]) \\ &\quad + g\bar{\psi}(x)[S(x) + i\gamma_5 P(x), \psi(x)] \}, \end{aligned} \quad (53)$$

which is apparently equal to the Dirac Lagrangian in (3) invariant under the Lorentz and gauge transformations. Equation (53) is written in components as

$$\begin{aligned} \mathcal{L}_D &= i\bar{\psi}^a(x)\gamma^\mu (\partial_\mu \psi^a(x) + gf^{abc}A_\mu^b(x)\psi^c(x)) \\ &\quad + igf^{abc}\bar{\psi}^a(x)(S^b(x) + i\gamma_5 P^b(x))\psi^c(x). \end{aligned} \quad (54)$$

4 Conclusions

We have reconstructed the $N = 2$ super Yang-Mills gauge theory in (2) and (3) by use of the noncommutative geometry developed by the present author. It is straightforward to reconstruct the $N = 4$ super Yang-Mills gauge theory in the same way as that in this paper. This subject was also picked up by Morita who assumed $\chi \wedge \chi = 0$ [12] and considered the term $F^0(x) = \langle \mathbf{d}, \mathcal{A}(x, y) \rangle + \langle \mathcal{A}(x, y), \mathcal{A}(x, y) \rangle$ instead of the potential term in (34). Therefore, his formulation is somewhat in the different context from that of our formulation.

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References

1. A. Connes, in The Interface of Mathematics and Particle Physics, eds. D. G. Quillen, G. B. Segal, and S. T. Tsou (Clarendon Press, Oxford, 1990) p.9. See also, Alain Connes and J. Lott, Nucl. Phys. **B**(Proc. Suppl.) **18B** (1990) 57.
2. References in the review work by J. Madore, J. Mourad, "Noncommutative Kaluza-Klein Theory", hep-th/9601169.
3. A.H.Chamseddine, G.Felder, J.Frölich, Phys. Lett. **B296** (1992) 109; Nucl. Phys. **B395** (1993) 672; A.H.Chamseddine, J.Frölich, Phys. Rev. **D50** (1994) 2893.
4. A.H.Chamseddine, Phys. Lett. , **B332** (1994) 349.
5. D. Kastler, Rev. Math. Phys. **5** (1993) 477; M. Dubois-Violette, Class. Quantum Grav. **6** (1989) 1709; R. Coquereaux, G. Esposito-Farese, G. Vaillant, Nucl. Phys. **B353**, (1991) 689; M. Dubois-Violette, R. Kerner, J. Madore, J. Math. Phys. **31** (1990) 316; B. Balakrishna, F. Gürsey, K.C. Wali, Phys. Lett. **B254** (1991),430; Phys. Rev. **D46** (1992) 6498.
6. R. Coquereaux, G. Esposito-Farése, G. Vaillant, Nucl. Phys. **B353** (1991) 689, R. Coquereaux, G. Esposito-Farese, F. Scheck, Int. Journ. Mod. Phys. **A7**(1992) 6555; R. Coquereaux, R. Haussling, N. Papadopoulos, F. Scheck, ibid. **7** (1992) 2809.
7. A. Sitarz, Phys. Lett. , **B308** (1993) 311, Jour. Geom. Phys. **15** (1995) 123.
8. Y. Okumura, Prog. Theor. Phys. **92** (1994) 625, K. Morita, Y. Okumura, Phys. Rev. **D50** (1994) 1016.
9. Y. Okumura, Phys. Rev. **D50** (1994) 1026, K. Morita, Y. Okumura, Prog. Theor. Phys. **91** (1994) 975.
10. Y. Okumura, Prog. Theor. Phys. **94** (1995) 589.
11. Y. Okumura, Prog. Theor. Phys. **95** (1996) 969.
12. K. Morita, Prog. Theor. Phys. **96** (1996) 787.